Name of College - S.S. College
J-Bad

Date - 18-06-2021
and 19-06-2021

Deptt - Mathematics
Topic - Tangent and Normal

Class - B.Sc (I) (Hons)
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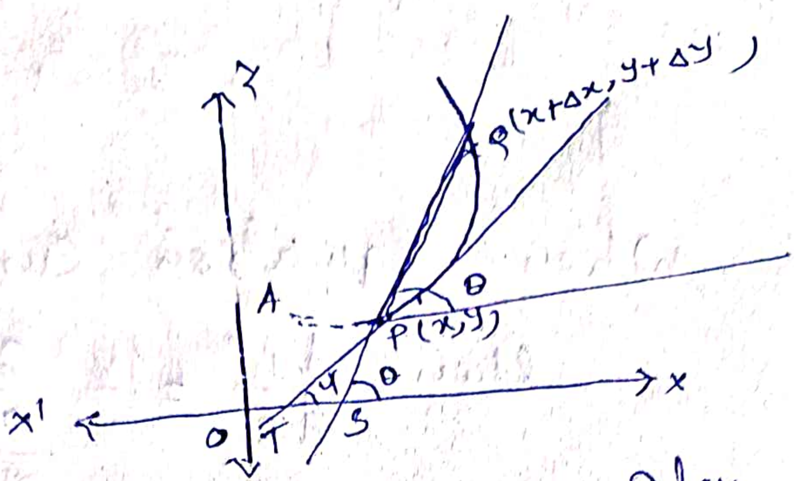
Study Material :→

Tangent -



Let P and Q be any two points on a given curve. When the point Q while moving along the curve coincides with the point P, then the limiting position of secant PQ is called the Tangent at the point P. P is called of contact.

A st. line perpendicular to tangent at P is called Normal to the Tangent. In this case P is called Foot of the Normal.

This Tangent is a st. line passing through two coincident points.

Note :→ 1. Tangent and Normals are mutual perpendicular st. line.

Slope of Tangent × slope of Normal = −1

2. $\frac{dy}{dx}$ stands for slope of Tangent at (x, y)

Equation of Tangent -

$$\frac{y-y_1}{x-x_1} = \left(\frac{dy}{dx}\right)_{x_1, y_1}$$

When the Equation of curve is $y = f(x)$

When Equation curve is $f(x, y) = 0$

then $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$$\therefore \frac{y-y_1}{x-x_1} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\Rightarrow (x-x_1)\frac{\partial f}{\partial x} + (y-y_1)\frac{\partial f}{\partial y} = 0$$

Equation of Normal →

If Equation of Curve is $y = f(x)$

then $\frac{y-y_1}{x-x_1}$ = slope of Normal

$$= -\frac{1}{\text{slope of Tangent}}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)_{x_1, y_1}}$$

$$(y-y_1)\frac{dy}{dx} = -(x-x_1)$$

$$\Rightarrow (x-x_1) + (y-y_1)\frac{dy}{dx} = 0$$

Equation of Normal when Equation of Curve is $f(x,y) = 0$

Equation of Normal is

$$(x-x_1) + (y-y_1)\frac{dy}{dx} = 0$$

$$\Rightarrow (x-x_1) - (y-y_1)\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = 0$$

$$\Rightarrow (x-x_1) = (y-y_1)\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\Rightarrow \frac{(x-x_1)}{\frac{\partial f}{\partial x}} = \frac{y-y_1}{\frac{\partial f}{\partial y}}$$

Remarks :→ 1. If Tangent Makes angle ψ with the +ve direction of x-axis

Then $\tan\psi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

2. If $P(x_1, y_1)$ be the point of contact. Then it must satisfies the
   (i) Equation of curve
   (ii) Equation of Tangent.
   (iii) Equation of Normal

12 Problem 1-

Find the Equation of the tangent and Normal at the point (α, β) of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution :→ Here Equation of the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(I)}$$

(α, β) be the point on the curve

$$\Rightarrow \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1 \qquad \text{(II)}$$

Equation of tangent at (α, β) is

$$\frac{y-\beta}{x-\alpha} = \left(\frac{dy}{dx}\right)_{\alpha, \beta} \qquad \text{(III)}$$

Equation of Normal at (α, β) is

$$\frac{y-\beta}{x-\alpha} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\alpha, \beta}} \qquad \text{(IV)}$$

Diff. (1) with respect to x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} = -\frac{y}{b^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}\frac{b^2}{a^2}$$

$$\left(\frac{dy}{dx}\right)_{\alpha,\beta} = -\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}$$

From (III) Equation of Tangent at (α,β) is

$$\frac{y-\beta}{x-\alpha} = -\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}$$

$$a^2\beta y - a^2\beta^2 = -b^2\alpha x + b^2\alpha^2$$

$$\Rightarrow \quad b^2\alpha x + a^2\beta y = a^2\beta^2 + b^2\alpha^2$$

$$\Rightarrow \quad \frac{x\alpha}{a^2} + \frac{\beta y}{b^2} = \frac{a^2\beta^2}{a^2b^2} + \frac{b^2\alpha^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}$$

using (II)

$$\Rightarrow \quad \frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1$$

From (IV).

Equation of Normal

$$\frac{y-\beta}{x-\alpha} = +\frac{1}{\frac{\alpha}{\beta}\cdot\frac{b^2}{a^2}}$$

$$\Rightarrow \quad (y-\beta)\alpha b^2 = (x-\alpha)\beta a^2$$

$$\Rightarrow \quad (x-\alpha)a^2\beta = (y-\beta)b^2\alpha$$

Problem 2. Find the Equation of the Tangent to the Curve

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

Parametric Equation of Curve is

$x = a\cos^3\theta \Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

$y = a\sin^3\theta. \Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta}.$$

Now Equation of Tangent at θ. [Applying $\frac{y-y_1}{x-x_1} = \left(\frac{dy}{dx}\right)_{x_1,y_1}$

Here $x_1 = a\cos^3\theta$
$y_1 = a\sin^3\theta$

$$\frac{y - a\sin^3\theta}{x - a\cos^3\theta} = -\frac{\sin\theta}{\cos\theta}$$

$\Rightarrow y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

$\Rightarrow x\sin\theta + y\cos\theta = a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$

$\Rightarrow x\sin\theta + y\cos\theta = a\sin\theta\cdot\cos\theta.$

Problem 3. Find the Equation of Tangent at (a, b) to the Curve

$$\left(\frac{x}{a}\right)^n + (y/b)^n = 2$$

solution :→ Here $x_1 = a$
$y_1 = b$

Equation of the Curve.

$$\left(\frac{x}{a}\right)^n + (y/b)^n = 2 \quad \text{—— (1)}$$

Since (a,b) lies on the curve

$\Rightarrow$ $\cancel{\left(\frac{a}{a}\right)^n + \left(\frac{b}{b}\right)^n} =$

Now Equation of Tangent at (a,b)

$$\frac{y-b}{x-a} = \left(\frac{dy}{dx}\right)_{a,b} \text{ ——— (II)}$$

Now Diff. (1) with respect to x

$$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\frac{dy}{dx} = 0$$

$$\frac{ny^{n-1}}{b^n}\frac{dy}{dx} = -\frac{nx^{n-1}}{a^n}$$

$$\therefore \frac{dy}{dx} = -\frac{b^n}{a^n}\frac{x^{n-1}}{y^{n-1}}$$

$$\left(\frac{dy}{dx}\right)_{a,b} = -\frac{b^n}{a^n}\cdot\frac{a^{n-1}}{b^{n-1}} = -\frac{b}{a}$$

$\therefore$ Equation of Tangent at (a,b) from (II)

$$\frac{y-b}{x-a} = -b/a$$

$\Rightarrow$ $ay - ab = -bx + ab$

$\Rightarrow$ $bx + ay = 2ab$

$\Rightarrow$ $\frac{x}{a} + y/b = 2$

Problem 4. Prove that $x/a + y/b = 1$ Touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

Solution.

By the question,

Equation of curve is

$$y = b e^{-x/a} \quad \text{—— (1)}$$

Since the curve crosses the axis of y

$\therefore$ Put $x = 0$ in (1)

$$y = b.$$

Thus point of contact is $(0, b)$

Now Diff. (1) with respect to x

$$\frac{dy}{dx} = -b e^{-x/a} \cdot \frac{1}{a}$$

$$\left(\frac{dy}{dx}\right)_{0,b} = -\frac{b}{a}$$

Now Equation of Tangent is

$$\frac{y-b}{x-0} = -b/a$$

[using $\frac{y-y_1}{x-x_1} = \left(\frac{dy}{dx}\right)_{x_1,y_1}$
Here $x_1 = 0$ $y_1 = b$.]

$\Rightarrow ay - ab = -bx$

$\Rightarrow bx + ay = ab$

$\Rightarrow x/a + y/b = 1$

Hence the Required Equation of Tangent

Problem :→ If the Normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle θ with the x-axis; show that its Equation is

Solution $y\cos\theta - x\sin\theta = a\cos 2\theta$.

Here Equation of the curve is:

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{———} \quad (I)$$

Now the Equation of Normal at (x_1, y_1) is

$$\frac{y - y_1}{\left(\frac{\partial f}{\partial y}\right)_{x_1, y_1}} = \frac{x - x_1}{\left(\frac{\partial f}{\partial x}\right)_{x_1, y_1}} \quad \text{———} \quad (II)$$

Here $f(x, y) = x^{2/3} + y^{2/3} - a^{2/3}$

$$\frac{\partial f}{\partial x} = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}.$$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{2/3 - 1} = \frac{2}{3} y^{-1/3} = \frac{2}{3y^{1/3}}.$$

$$\therefore \left(\frac{\partial f}{\partial x}\right)_{x_1, y_1} = \frac{2}{3 x_1^{1/3}}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x_1, y_1} = \frac{2}{3 y_1^{1/3}}.$$

∴ Equation of Normal.

$$\cancel{\frac{3}{2}} \frac{x - x_1}{x_1^{1/3}} = \cancel{\frac{3}{2}} \frac{y - y_1}{y_1^{1/3}}$$

$$\Rightarrow y - y_1 = \frac{x_1^{1/3}}{y_1^{1/3}}(x - x_1) \quad \text{——— (8)}$$

Which ~~can be~~ have per-in the form $y = mx + c$.

Since (11) makes angle θ with the x-axis, therefore

$$\tan\theta = \frac{x^{1/3}}{y^{1/3}}$$

$$\Rightarrow \frac{x_1^{1/3}}{\sin\theta} = \frac{y_1^{1/3}}{\cos\theta} = k$$

$$\therefore x_1^{1/3} = k\sin\theta \Rightarrow x_1^{2/3} = k^2\sin^2\theta.$$

$$y_1^{1/3} = k\cos\theta \Rightarrow y_1^{2/3} = k^2\cos^2\theta$$

$$\therefore x_1^{2/3} + y_1^{2/3} = k^2$$

$$\therefore k = \sqrt{x_1^{2/3} + y_1^{2/3}}$$

$$\frac{x_1^{1/3}}{\sin\theta} = \frac{y_1^{1/3}}{\cos\theta} = \frac{\sqrt{x_1^{2/3} + y_1^{2/3}}}{\sqrt{\sin^2\theta + \cos^2\theta}} = a$$

$$x_1^{1/3} = a^{1/3}\sin\theta \Rightarrow x_1 = a\sin^3\theta$$

$$y_1^{1/3} = a^{1/3}\cos\theta \Rightarrow y_1 = a\cos^3\theta$$

$\therefore$ Equation of Normal

$y - a\cos^3\theta = (x - a\sin^3\theta)$ $\dfrac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}$.

$\Rightarrow y\cos\theta - a\cos^4\theta = x\sin\theta - a\sin^4\theta$.

$\Rightarrow x\sin\theta - y\cos\theta = a\sin^4\theta - a\cos^4\theta$.

$= -a[\cos^4\theta - \sin^4\theta]$

$= -a[\cos^2\theta - \sin^2\theta][\cos^2\theta + \sin^2\theta]$

$= -a\cos 2\theta$.

$\therefore y\cos\theta - x\sin\theta = a\cos 2\theta$.

is the required Equation of Normal

Problem :→ Show that the sum of the intercepts of the tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ upon the co-ordinate axes is constant.

Solution Let (x_1, y_1) be any point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

The Equation of tangent at (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{x_1, y_1}$$

here. $\sqrt{x} + \sqrt{y} = \sqrt{9}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\sqrt{y/x}$$

$$\left(\frac{dy}{dx}\right)_{x_1, y_1} = -\sqrt{\frac{y_1}{x_1}}$$

$\therefore$ Equation of Tangent at (x_1, y_1) is

$$y - y_1 = -(x - x_1)\sqrt{\frac{y_1}{x_1}}$$

$$y\sqrt{x_1} - y_1\sqrt{x_1} = -x\sqrt{y_1} + x_1\sqrt{y_1}$$

$$\Rightarrow x\sqrt{y_1} + y\sqrt{x_1} = x_1\sqrt{y_1} + y_1\sqrt{x_1}$$

$$= \sqrt{x_1 y_1}\left(\sqrt{x_1} + \sqrt{y_1}\right)$$

$$= \sqrt{x_1 y_1}\sqrt{9}$$

$\therefore$ Equation of Tangent at (x_1, y_1)

is $\sqrt{y_1}\,x + \sqrt{x_1}\,y = \sqrt{x_1 y_1}\sqrt{9}$

Now for intercept on x-axis

put $y = 0$.

$$\sqrt{y_1}\, X = \sqrt{x_1 y_1}\,\sqrt{a}$$

$$X = \sqrt{x_1}\sqrt{a}$$

Similary for intercept on y-axis

$x = 0$

$$y\sqrt{x_1} = \sqrt{x_1 y_1}\,\sqrt{a}$$

$$y = \sqrt{y_1}\,\sqrt{a}$$

$\therefore$ Sum of the intercept on axes.

$$\sqrt{x_1}\sqrt{a} + \sqrt{y_1}\sqrt{a}$$

$$= \sqrt{a}\,(\sqrt{x_1} + \sqrt{y_1})$$

$$= \sqrt{a}\,.\,\sqrt{a}$$

$$= a \quad = \text{const ans.}$$

Hence the result.